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Comparative statics and sign indeterminacy in a simple neoclassical macroeconomic model

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Abstract: In this paper, we analyse a simple two-period neoclassical macroeconomic model —short and long term— that exclusively considers the real sector of the economy (labour and goods markets). It is shown how, under a general characterization, some important signs of comparative statics are undetermined. This ambiguity is a consequence of the ubiquity of the real interest rate tying intertemporally the four markets considered. By imposing simplifying assumptions, the signs are determined at the cost of losing both generality and empirical adequacy. This fact limits the empirical relevance of a large part of the models commonly used in teaching macroeconomics, where ambivalent results are avoided because of the need for clear answers on the effects of fiscal and monetary policy interventions. Taking a positive view, these results compels us to take general interdependence seriously and to pay more attention to the complete set of theoretical possibilities that arise when modelling macroeconomic systems.

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Resumen: En este trabajo se analiza un modelo macroeconómico neoclásico sencillo de dos periodos —corto y largo plazo— que considera exclusivamente el sector real de la economía (mercados de bienes y de trabajo). Se comprueba cómo, bajo una caracterización general, algunos signos importantes de la estática comparativa están indeterminados. Esta ambigüedad está causada por la presencia del tipo de interés real que liga intertemporalmente los cuatro mercados considerados. Añadiendo supuestos simplificadores se consigue determinar los signos al coste de una pérdida de generalidad y admisibilidad empírica. Este hecho limita considerablemente la relevancia empírica de buena parte de los modelos utilizados habitualmente para la enseñanza de la macroeconomía, donde se evitan resultados de este tipo, dado que se persigue fundamentalmente responder a preguntas sobre el sentido de las intervenciones de política fiscal y monetaria. Leído positivamente, este resultado obliga a prestar una mayor atención a todas las posibilidades teóricas que surgen al modelizar sistemas de interdependencia general, como son los propios de la macroeconomía.

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1. Introduction

In many universities, the teaching of macroeconomics is based on popular textbooks (e.g., Blanchard; Abel & Bernanke; Dornbusch, Fischer & Starz; Froyen; Gordon; Mankiw) in which the IS-LM model, presented by Hicks (1937) 80 years ago, is central. This model represents a static equilibrium of both goods and money markets in which production may vary freely to a given price. Being this assumption a rough approach for short run analysis, it must be relaxed to include a more realistic behaviour concerning inflation. For this, it is assumed that, in the medium term, prices respond to demand shocks and that this response depends on the supply side of the economy. Consequentially, building on this model, a supply side is added, which provides the tone predominantly neoclassical or neo-Keynesian of a model, already complete, in the medium term. For its part, the working of the economy in the long term is assumed to be rather different. The long term is governed by a strict neoclassical supply side in which money is neutral or superneutral and full employment prevails. At least for pedagogical purposes, the Solow growth equation would acceptably represent such a long term behaviour. The Solow model consists of a single dynamic equation that represents the adjustment process from a situation of non-zero per capita net investment to a steady state in which per capita capital remains constant. Although used to account for a set of stylized facts of growth (truly result of the general interdependence of many markets), the Solow model is a partial equilibrium model of the goods market without an explicit interest rate.

This usual approach to teaching macroeconomics changed from the 80s onwards. Both the renewal of the growth theory in the 1980s and the neo-neoclassical restoration (Monetarism, New Classical Economy, Real Business Cycle theory) led to varying the importance of the pieces of the typical macroeconomic model or the order of presentation of the issues. In contrast to what was being done before this time, to explain the growth phenomenon as a starting point is now common (e.g., Abel & Bernanke; Barro; Burda & Wyplosz; Dornbusch et al.; Jones; Hall & Pappell; Mankiw; Romer; Sørensen & Whitta-Jacobsen). After explaining growth, textbooks proceed to analyse the short or the medium term, in which the dynamics is typically New Keynesian: equations of sticky wages and prices and Phillips curves based on the non-accelerating inflation rate of unemployment (NAIRU). Under this characterization, money is not neutral; otherwise, monetary policy not only would be inefficient, as in the New Classical Economy, but irrelevant, as conceived by the Real Business Cycle theory.

In addition to giving greater weight to economic growth and some priority over the business cycle dynamics, the need to root macroeconomic behavioral relations in microfoundations has also been widely introduced in textbooks. Nevertheless, although explained to a certain extent in intermediate textbooks, a more rigorous microfoundation is left for higher-level textbooks (e.g., Blanchard & Fischer; Chugh; Romer; Sørensen & Whitta-Jacobsen). A microfounded long term model is the Ramsey-Cass-Koopmans model of optimal growth, which has finally become the reference model for the long term. Unlike the Solow-Swan model, this model is explicitly microfounded because the saving rate is the result of an optimal intertemporal consumption plan. Although it solves one of the problems of the Solow's model that is its backward looking dynamics, it remains a partial equilibrium model of the goods market. However, even the textbooks insisting on the importance of microfoundations cannot avoid landing in the IS-LM model enhanced with a New Keynesian supply side to explain the short and medium term dynamics, i.e., the cyclical fluctuations and the monetary non-neutrality as its main cause.

In the shift in emphasis from the aggregates to the microfoundations of macroeconomics, Fisher's model of intertemporal choice (1907)[1] is central for analysing decisions which are truly intertemporal (consumption, saving, investment, indebtedness). In microfounded macroeconomics, the zero-degree homogeneity of demand functions makes monetary prices irrelevant, whereas relative prices are not affected. Additionally, in this context, the problem of how to ensure monetary non-neutrality always exists (and monetary non-neutrality seems to be very relevant on empirical grounds) and, therefore, a gap between the short and medium term (New Keynesian) and the long term (neoclassical) continues to be open.

An attempt to obtain cyclical oscillations in the short and medium term without appealing to the monetary side of the economy, which seems to have a bad fit in microfounded macroeconomics, can be found in the Real Business Cycle models. These, specially their extreme versions lacking money, place the cause of the cyclical oscillations in productivity shocks. The presence of intertemporal links in the real sector of the economy do the rest. But, when intertemporal links are explicitly considered, it cannot be ruled out that anything can occur as we show in this work. Precisely the insistence of the Real Business Cycle modelling on calibration (to particularize the model by means of certain parametrization) is the way to avoid such indeterminacies.

In this study, we analyse a simple neoclassical macroeconomic model of two periods, equivalent to the short and the long terms, which only considers the real sector of the economy (goods and labour markets). Due to the general characterization undertaken, it is verified that some of the important signs of comparative statics are undefined. This ambiguity is due to the real interest rate that intertemporally links the four markets considered. By imposing simplifying assumptions (exogenous labour supply functions[2] and independent factors of production), the signs

[1] Fisher's intertemporal analysis was anticipated in 1834 by John Rae (*New Principles of Political Economy*), to whom Fisher dedicated *The Rate of Interest*, and by Eugen von Böhm-Bawerk. See Geanakoplos (2007).

[2] Two-period partial models can be found in, e.g., Barro (2008), Chugh (2015), and Williamson (2014). In regard to the presence of the interest rate in the labour supply as a result of using intertemporal choice models with labour markets, Garín *et al*. (2016, p. 2) claim that "our experience suggested that the intertemporal supply relationship (due to an effect of the real interest rate on labour supply), which is the hallmark of the Williamson (2014) approach, was ultimately confusing to students [...] We have simplified this by assuming that the labour supply does not depend on the real interest rate. This can be formally motivated through the use of the preferences proposed in Greenwood, Hercowitz, and Huffman (1988), which feature no wealth effect on the labour supply."

can be determined at the expense of a loss of generality and empirical admissibility. This fact greatly limits the empirical relevance of most of the models commonly used to teach macroeconomics, in which similar assumptions are made to avoid these complications. For example, in the typical IS curve, the consumption function does not depend on the real wage (current or future) or on the level of employment, nor does it depend on any component of wealth; the investment function implicitly assumes independent production factors when the prices of other factors of production -e.g., the real wage- are not included as arguments. Similarly, on the supply side, the interest rate does not influence either the labour demand or supply. In a positive sense, from the result obtained from this model, the need for paying greater attention to the various theoretical possibilities that arise when modelling systems of general interdependence is derived.

The rest of this paper is organized as follows. In section 2, the decision functions of households are discussed: consumption and labour supply functions, both current and future. Although elementary by assuming homogeneous individuals, the aggregation performed enables us to consider the population, current and future, as an explicit component of wealth and its influence on the corresponding aggregates. The labour demands and product supplies for firms in both periods, as well as the investment demand, are discussed in Section 3. In section 4, public sector accounts are consolidated through the Ricardian equivalence principle. Section 5 presents the equilibrium of the system and solves the vector of prices through a linear approximation. The indeterminacy of signs due to the opposite effects of the real interest rate on certain variables is verified. In section 6, the interactions provoking this indeterminacy are eliminated by assuming exogenous labour supplies and independent production factors. Section 7 analyses the comparative statics of technological improvements in both periods following the Real Business Cycle approach. Once again, an ambiguity appears in one of the signs, that which corresponds to the interest rate. Finally, Section 8 presents the conclusions.

2. Decisions of the households

For a household (later, the households will be aggregated; until then, the subscript *i* will not be used in order to avoid complicating the notation), the available resources at the start of $t=0$ are:

i) b_0: the net holdings of financial assets other than stocks; it may be that $b_0<0$ for the case of net indebtedness; and
ii) $v_0\geq 0$: the holdings of stocks or other financial assets representing rights on firms.

Additionally, there are those obtained during $t=0$:

iii) $w_0 n_0\geq 0$: labour income. The labour supply, n_0, is a decision variable. Each household is endowed with a time unit such that $n_0+l_0=1$, with l_0 as the leisure time.

Except for taxes, the uses at $t=0$ are decision variables:

i) $p_0c_0\geq 0$: consumption;
ii) $p_0t_0\geq 0$: fixed taxes;
iii) b_1: the net demand of financial assets other than stocks; and
iv) $v_1\geq 0$: the demand for stocks.

Due to the absence of risk, b_1 and v_1 are perfect substitutes, yield the same interest rate, and are jointly demanded.

The budget constraint at $t=0$ is:

$$b_0+v_0+w_0n_0\geq b_1+v_1+p_0c_0+p_0t_0 \tag{1}$$

The saving is the unconsumed disposable income, $p_0s_0=w_0n_0-p_0t_0-p_0c_0$, which is placed in financial assets and may eventually be negative. Thus, it is satisfied that $p_0s_0=b_1-b_0+v_1-v_0$.

The resources at $t=1$ are:

i) b_1: the net holdings of assets received from the previous period;
ii) $v_1\geq 0$: the holding of stocks received from the previous period;
iii) $i(b_1+v_1)$: the yield or interest payment for financial investments made in the previous period; and
iv) $w_1n_1\geq 0$: labour income. It is satisfied that $n_1+l_1=1$. As at $t=0$, the labour supply is a decision variable.

The uses at $t=1$ are:

i) $p_1c_1\geq 0$: the consumption; and
ii) $p_1t_1\geq 0$: the fixed taxes.

Thus, the budget constraint at $t=1$ is:

$$(1+i)(b_1+v_1)+w_1n_1\geq p_1c_1+p_1t_1 \tag{2}$$

At $t=1$, no legacy of any sign is left: at the optimum, the accounts are settled. It is therefore satisfied that the "dissaving" $p_1s_1=w_1n_1+i(b_1+v_1)-p_1t_1-p_1c_1=-(b_1+v_1)$, i.e., the accumulated assets are liquidated and consumed.

The intertemporal budget constraint is obtained by replacing (1) in (2) and rearranging it, which leads to $(1+i)(b_0+v_0+w_0n_0-p_0t_0-p_0c_0)+w_1n_1\geq p_1c_1+p_1t_1$.
Expressing it in monetary units of $t=0$, we have the following:

$$b_0 + v_0 + w_0 n_0 + \frac{w_1 n_1}{1+i} \geq p_0 c_0 + \frac{p_1 c_1}{1+i} + p_0 t_0 + \frac{p_1 t_1}{1+i}.$$

Defining the real interest rate as $1+r = \frac{1+i}{1+\pi}$, with $p_1 = p_0(1+\pi)$ resulting from inflation, and dividing by p_0, we can express the constraint in units of product of $t=0$ as $\frac{b_0+v_0}{p_0} + \frac{w_0}{p_0} n_0 + \frac{1}{1+r}\frac{w_1}{p_1} n_1 - t_0 - \frac{t_1}{1+r} \geq c_0 + \frac{c_1}{1+r}$.

Replacing the labour supply, n_t, with the quantity of leisure demanded, l_t, results in the following:

$$\frac{b_0+v_0}{p_0} + \frac{w_0}{p_0} - t_0 + \frac{1}{1+r}\left(\frac{w_1}{p_1} - t_1\right) \geq \frac{w_0}{p_0} l_0 + \frac{1}{1+r}\frac{w_1}{p_1} l_1 + c_0 + \frac{c_1}{1+r} \tag{3}$$

This is the budget constraint when all available time is not spent working ($n_0,\ n_1 < 1$) but, instead, part of it is spent demanding leisure time ($l_0,\ l_1 > 0$). The real wealth of a household is defined as the sum of its financial holdings plus the present value of the wage earnings after taxes, under the assumption that no leisure time is demanded, i.e., considering the full labour potential. It depends on the three relative prices of the model, $r, \frac{w_0}{p_0}$ and $\frac{w_1}{p_1}$, and of the real taxes t_0 and t_1. So

$$W\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; t_0, t_1\right) = \frac{b_0+v_0}{p_0} + \frac{w_0}{p_0} - t_0 + \frac{1}{1+r}\left(\frac{w_1}{p_1} - t_1\right) \tag{4}$$

Under some simple additive-type utility function (see Appendix A.1), consumption, leisure, and saving functions are obtained, whose signs are those frequently assumed by standard macroeconomics models. Thus, an increase in the real interest rate reduces the demand of current goods - both consumption and leisure- and increases the demand of future goods. Consequently, the effects on the labour supply are the opposite: when the real interest rate increases, the current labour supply increases, and the future labour supply decreases. An increase in any of the two real wages increases consumption in both periods because it causes a wealth effect. The effect of the real wage on the labour supply goes in the same direction with respect to the labour supply of the same period but in the opposite with respect to the other period: intertemporal substitution consist in working more time when the real wage is higher. Saving depends positively on both the real interest rate and the current real wage, but negatively on the future real wage because of its positive wealth effect on current consumption.

To analyse the effects of population levels on the aggregates, it is necessary to aggregate for all households (these are subindexed by i). From equation (4), the real wealth is:

$$\mathbf{W} = \sum_i W_i = \frac{B_0+V_0}{p_0} + \left(\frac{w_0}{p_0} - t_0\right)\mathbf{N_0} + \frac{1}{1+r}\left(\frac{w_1}{p_1} - t_1\right)\mathbf{N_1} \tag{5}$$

where:

i) $B_0 = \sum_i b_{i,0}$: with possible compensations for loans between households, such that the aggregate B_0 represents a net holding of claims on the public sector;

ii) $V_0 = \sum_i v_{i,0}$: the stocks held by households;

iii) $\mathbf{N_0}$ and $\mathbf{N_1}$ are the maximum amounts of work that could be available if households do not demand leisure time. These can be identified with the potentially active populations at $t=0$ and $t=1$; and

iv) $T_0 = \sum_i t_{i,0}$ and $T_1 = \sum_i t_{i,1}$ are the tax collections in both periods. If a poll tax is assumed, then $T_0 = \sum_i t_{i,0} = \mathbf{N_0} t_0$ and $T_1 = \sum_i t_{i,1} = \mathbf{N_1} t_1$.

Under the utility function assumed in the Appendix A.1, the aggregate consumption functions are $C_0 = \sum_i c_{i,0} = \sum_i k_c W_i = k_c \mathbf{W}$ and $C_1 = \sum_i c_{i,1} = \sum_i k_c \beta(1+r) W_i = k_c \beta(1+r)\mathbf{W}$, where $0 < \beta < 1$ is the discount factor, $k_c = \frac{1}{1+\psi+\beta(1+\psi)}$, and $\psi = \frac{\partial U}{\partial \ln l_0}$. We have obtained the wealth-dependent consumption function (consumption as permanent income, Friedman, 1957). The aggregate labour supply functions for $t=0,\ 1$ are $N_0^s = \sum_i n_{i,0} = \sum_i (1-l_{i,0}) = \mathbf{N_0} - k_c \psi \frac{p_0}{w_0}\mathbf{W}$ and $N_1^s = \sum_i n_{i,1} = \sum_i (1-l_{i,1}) = \mathbf{N_1} - k_c \beta \psi \frac{p_1}{w_1}(1+r)\mathbf{W}$. That is, they are the aggregate labour potentials not allocated to leisure. It is verified that aggregate wealth is allocated between consumption and leisure just as individual wealth is assigned, $\mathbf{W} = k_c(1+\psi+\beta(1+\psi))\mathbf{W}$, with $k_c\psi$, $k_c\beta\psi$, k_c, and $k_c\beta$ being the percentages of aggregate wealth assigned to current and future leisure, and current and future consumption, respectively, that add up to the unit. That is, wealth (physical and human capital) is optimally allocated in its four available uses.

The signs of the aggregate functions retain those of the individual functions. The population operates as a scale variable at the same period of the real wage, but it is the future population for the real interest rate because it acts discounting future labour potential:

$$\frac{\partial C_0}{\partial r}=\mathbf{N}_1\frac{\partial c_0}{\partial r}<0;\ \frac{\partial C_0}{\partial\left(\frac{w_0}{p_0}\right)}=\mathbf{N_0}\frac{\partial c_0}{\partial\left(\frac{w_0}{p_0}\right)}>0;\ \frac{\partial C_0}{\partial\left(\frac{w_1}{p_1}\right)}=\mathbf{N}_1\frac{\partial c_0}{\partial\left(\frac{w_1}{p_1}\right)}>0;$$

$$\frac{\partial C_1}{\partial r}=\mathbf{N_1}\frac{\partial c_1}{\partial r}>0;\ \frac{\partial C_1}{\partial\left(\frac{w_0}{p_0}\right)}=\mathbf{N}_0\frac{\partial c_1}{\partial\left(\frac{w_0}{p_0}\right)}>0;\ \frac{\partial C_1}{\partial\left(\frac{w_1}{p_1}\right)}=\mathbf{N_1}\frac{\partial c_1}{\partial\left(\frac{w_1}{p_1}\right)}>0.$$

Regarding the effects of the population size changes, we have the following:

$$\frac{\partial C_0}{\partial\mathbf{N_0}}=k_c\left(\frac{w_0}{p_0}-t_0\right)>0;\ \frac{\partial C_0}{\partial\mathbf{N_1}}=k_c\frac{1}{1+r}\left(\frac{w_1}{p_1}-t_1\right)>0;$$

$$\frac{\partial C_1}{\partial\mathbf{N_0}}=k_c\beta(1+r)\left(\frac{w_0}{p_0}-t_0\right)>0;\ \frac{\partial C_1}{\partial\mathbf{N_1}}=k_c\beta\left(\frac{w_1}{p_1}-t_1\right)>0.$$

That is, the wealth effects derived from population changes are distributed in both periods. The effects on the aggregate labour supplies of the real wage and the real interest rate variations are:

$$\frac{\partial N_0^s}{\partial r}=-\mathbf{N}_1\frac{\partial l_0}{\partial r}>0;\ \frac{\partial N_0^s}{\partial\left(\frac{w_0}{p_0}\right)}=-\mathbf{N_0}\frac{\partial l_0}{\partial\left(\frac{w_0}{p_0}\right)}>0;\ \frac{\partial N_0^s}{\partial\left(\frac{w_1}{p_1}\right)}=-\mathbf{N}_1\frac{\partial l_0}{\partial\left(\frac{w_1}{p_1}\right)}<0;$$

$$\frac{\partial N_1^s}{\partial r}=-\mathbf{N_1}\frac{\partial l_1}{\partial r}<0;\ \frac{\partial N_1^s}{\partial\left(\frac{w_0}{p_0}\right)}=-\mathbf{N}_0\frac{\partial l_1}{\partial\left(\frac{w_0}{p_0}\right)}<0;\ \frac{\partial N_1^s}{\partial\left(\frac{w_1}{p_1}\right)}=-\mathbf{N_1}\frac{\partial l_1}{\partial\left(\frac{w_1}{p_1}\right)}>0.$$

Intertemporal substitution leads the households to supply more labour when the real wage is higher. Furthermore an increase in the real interest rate increases current labour supply and reduces future labour supply due to its opposite effects on the respective leisure demands.

The effects of population changes on labour supplies are:

$$\frac{\partial N_0^s}{\partial\mathbf{N_0}}=1-k_c\psi\frac{p_0}{w_0}\left(\frac{w_0}{p_0}-t_0\right)>0\ ?\ ;\ \frac{\partial N_0^s}{\partial\mathbf{N_1}}=-k_c\psi\frac{1}{1+r}\frac{p_0}{w_0}\left(\frac{w_1}{p_1}-t_1\right)<0;$$

$$\frac{\partial N_1^s}{\partial\mathbf{N_0}}=-k_c\beta\psi(1+r)\frac{p_1}{w_1}\left(\frac{w_0}{p_0}-t_0\right)<0;\ \frac{\partial N_1^s}{\partial\mathbf{N_1}}=1-k_c\beta\psi\frac{p_1}{w_1}\left(\frac{w_1}{p_1}-t_1\right)>0\ \mathbf{?}$$

where it is verified that an increase in the potentially active population in a period may increase the labour supply in that period (we assume that this will occur, even though the sign is ambiguous; see Cendejas 2016), albeit in a measure less than proportional. And it certainly reduces it in the other period because increases in population cause wealth effects that increase the demand for leisure.

The aggregate saving function is:

$$S_0=\frac{w_0}{p_0}N_0^s-\mathbf{N_0}t_0-C_0=\frac{w_0}{p_0}\mathbf{N_0}-\frac{w_0}{p_0}k_c\psi\frac{p_0}{w_0}\mathbf{W}-\mathbf{N_0}t_0-k_c\mathbf{W}=\left(\frac{w_0}{p_0}-t_0\right)\mathbf{N_0}-k_c(1+\psi)\mathbf{W}$$

whose signs are:

$$\frac{\partial S_0}{\partial r}=\mathbf{N_1}k_c(1+\psi)\frac{1}{(1+r)^2}\left(\frac{w_1}{p_1}-t_1\right)>0;\quad \frac{\partial S_0}{\partial\left(\frac{w_0}{p_0}\right)}=\mathbf{N_0}(1-k_c(1+\psi))>0;\quad \frac{\partial S_0}{\partial\left(\frac{w_1}{p_1}\right)}=-\mathbf{N_1}k_c(1+\psi)\frac{1}{1+r}<0;$$

$$\frac{\partial S_0}{\partial \mathbf{N_0}}=\left(\frac{w_0}{p_0}-t_0\right)(1-k_c(1+\psi))>0;\quad \frac{\partial S_0}{\partial \mathbf{N}_1}=-k_c(1+\psi)\frac{1}{1+r}\left(\frac{w_1}{p_1}-t_1\right)<0.$$

In general, any variation that increases current consumption reduces saving. When there are both a substitution effect and income and wealth effects that can eventually compensate each other, the signs found here can also be obtained from other functional forms, if the necessary assumptions for the predominance of the sign of the substitution effect are met (Cendejas, 2016). Summarizing, we have that

$$C_0=C_0\left(\underset{-}{r},\underset{+}{\frac{w_0}{p_0}},\underset{+}{\frac{w_1}{p_1}};\underset{+}{\mathbf{N_0}},\underset{+}{\mathbf{N_1}},\underset{-}{t_0},\underset{-}{t_1}\right);\;\; S_0=S_0\left(\underset{+}{r},\underset{+}{\frac{w_0}{p_0}},\underset{-}{\frac{w_1}{p_1}};\underset{+}{\mathbf{N_0}},\underset{-}{\mathbf{N_1}},\underset{-}{t_0},\underset{+}{t_1}\right);$$

$$N_0^s=N_0^s\left(\underset{+}{r},\underset{+}{\frac{w_0}{p_0}},\underset{-}{\frac{w_1}{p_1}};\underset{+}{\mathbf{N_0}},\underset{-}{\mathbf{N_1}},\underset{-}{t_0},\underset{+}{t_1}\right);\;\; N_1^s=N_1^s\left(\underset{-}{r},\underset{-}{\frac{w_0}{p_0}},\underset{+}{\frac{w_1}{p_1}};\underset{-}{\mathbf{N_0}},\underset{+}{\mathbf{N_1}},\underset{+}{t_0},\underset{-}{t_1}\right).$$

3. The public sector

We assume that the public sector makes its decisions exogenously and is financed under competitive conditions at the market interest rate. Thus, at t=0, we have that $\frac{B_0}{p_0}+G_0=T_0+\frac{B_1}{p_1}$, i.e., resources coming from taxes and the issuance of new debt are allocated to pay off the outstanding debt and to public spending. At *t=1*, debt B_1 is remunerated and redeemed while, once again, more is spent and collected, verifying that $\frac{B_1}{p_1}(1+r)+G_1-T_1=0$. Replacing $\frac{B_1}{p_1}$ with $\frac{B_0}{p_0}+G_0-T_0$, the intertemporal budget constraint of the public sector is obtained:

$$T_0+\frac{T_1}{1+r}\geq\frac{B_0}{p_0}+G_0+\frac{G_1}{1+r} \tag{6}$$

In short, the outstanding debt balance will be redeemed with fiscal surpluses. The wealth of households is modified because these consider the taxes that they will have to pay to redeem the debt and finance public spending. Replacing (6) in (5)

$$\mathbf{W}=\frac{B_0+V_0}{p_0}+\left(\frac{w_0}{p_0}-t_0\right)\mathbf{N_0}+\frac{1}{1+r}\left(\frac{w_1}{p_1}-t_1\right)\mathbf{N_1}=\frac{B_0+V_0}{p_0}+\frac{w_0}{p_0}\mathbf{N_0}+\frac{1}{1+r}\frac{w_1}{p_1}\mathbf{N_1}-T_0-\frac{T_1}{1+r}=\ldots$$

$$\ldots=\left(T_0-G_0+\frac{T_1}{1+r}-\frac{G_1}{1+r}\right)+\frac{V_0}{p_0}+\frac{w_0}{p_0}\mathbf{N_0}+\frac{1}{1+r}\frac{w_1}{p_1}\mathbf{N_1}-T_0-\frac{T_1}{1+r}$$

leads to the following:

$$\mathbf{W}=\frac{V_0}{p_0}+\frac{w_0}{p_0}\mathbf{N_0}+\frac{1}{1+r}\frac{w_1}{p_1}\mathbf{N_1}-G_0-\frac{G_1}{1+r} \tag{7}$$

The Ricardian equivalence is verified, which consists of the irrelevance of how the public sector is financed: only the amount of public spending is relevant because neither debt nor taxes are included in the budget constraint (Barro, 1974).

The consumption, saving and labour supply functions in both periods, written as a function of the levels of public spending and not as a function of taxes, maintain the same signs.

4. Decisions of the firms

The production function, $y = f(k,n)$, satisfies the following:

i) $f_k^{'} > 0$ and $f_n^{'} > 0$: the marginal productivities are positive;

ii) $y = f(k,n)$ is differentiable at least twice (C^2 class);

iii) the equality of the second cross derivatives, $f_{kn}^{"} = f_{nk}^{"}$, which determines the symmetry of the cross-effects in the demands of production factors;

iv) the need for, at least, one factor to produce: $f(0,0) = 0$; and

v) it is fulfilled that $\left|Hf(k,n)\right| = \begin{vmatrix} f_{kk}^{"} & f_{kn}^{"} \\ f_{kn}^{"} & f_{nn}^{"} \end{vmatrix} = f_{kk}^{"} f_{nn}^{"} - f_{kn}^{"2} > 0$ and its minors change signs, thus $f_{kk}^{"} < 0$ $f_{nn}^{"} < 0$ (strict concavity).

The representative firm takes prices as given as if they were determined in perfectly competitive product and factors markets. It chooses a production plan that maximizes the discounted flow of benefits. In the short term, the optimal amount of capital cannot be chosen, and therefore at $t = 0$, $\overline{k}_0$ is given. For $t = 1$, the long term, product prices and factors are unknown, and consequently, the production plan is conditioned and is optimal for a certain vector of expectations. In accordance to a certain expectation, to achieve the optimal capital at $t = 1$, it is necessary to invest at $t = 0$, being the gross investment function $I_0 = k_1 - \overline{k}_0 + \delta\overline{k}_0 = k_1 - (1-\delta)\overline{k}_0$.

Under these assumptions, when a firm maximizes its present value (see Appendix A.2) the usual signs appear: labour demands depend negatively on the real wage of the same period but they do not on the other period wage, and the demand of capital depends negatively on the real interest rate. Under complementarity of the production factors, $\dfrac{\partial k_1}{\partial\left(\dfrac{w_1}{p_1}\right)} = \dfrac{\partial n_1}{\partial r} < 0$, that is, the future real wage rate affects current investment and the real interest rate affects future labour demand. This intertemporal link is commonly ignored when the investment function is assumed to depend only on the interest rate. This would be correct if the factors of production were independent and consequently, there were not a cross-effect in prices. Regarding product supplies, current product supply depends negatively on the current real wage, and future product supply does it negatively on both the future real wage and the real interest rate. These latter signs are not affected by the complementarity or independency assumptions made on the production function.

There exists an aggregate production function under both homogeneity of degree one of the individual production functions and competitive assumptions (Sargent, 1979, Chap. 1). In this case, the theorem of Euler is fulfilled and, by aggregating for all firms (subindexed by j) we have that

$$Y = \sum_j y_j = \sum_j \left(f_k^{'} k_j + f_n^{'} n_j \right) = \sum_j \left((r+\delta)k_j + \frac{w}{p} n_j \right) = (r+\delta)\sum_j k_j + \frac{w}{p}\sum_j n_j = (r+\delta)K + \frac{w}{p}N$$

which refers to a function $Y = F(K,N)$ that is homogeneous of degree 1 as well. The aggregation for all firms does not change signs. For $t=1$, we have that $K_1 = \sum_j k_{j,1}\left(\frac{w_1}{p_1}, r\right) = K_1\left(\underset{-}{\frac{w_1}{p_1}}, \underset{-}{r}\right)$, $N_1^d = \sum_j n_{j,1}\left(\frac{w_1}{p_1}, r\right) = N_1^d\left(\underset{-}{\frac{w_1}{p_1}}, \underset{-}{r}\right)$ and $Y_1^s = \sum_j y_{j,1}\left(\frac{w_1}{p_1}, r\right) = Y_1^s\left(\underset{-}{\frac{w_1}{p_1}}, \underset{-}{r}\right)$. Additionally, for $t=0$, $I = K_1 - (1-\delta)K_0 = K_1\left(\frac{w_1}{p_1}, r\right) - (1-\delta)\sum_j k_{j,0} = I\left(\underset{-}{\frac{w_1}{p_1}}, \underset{-}{r}\right)$;

$N_0^d = \sum_j n_{j,0}\left(\frac{w_0}{p_0}\right) = N_0^d\left(\underset{-}{\frac{w_0}{p_0}}\right)$, and $Y_0^s = \sum_j y_{j,0}\left(\frac{w_0}{p_0}\right) = Y_0^s\left(\underset{-}{\frac{w_0}{p_0}}\right)$. As already stated, investment depends on the future real wage with negative sign if complementarity of the factors is assumed. It would not depend on it if factors were independent.

In the aggregate and under exhaustion of the product in both periods, the value of the firm is:

$$\frac{V_0}{p_0} = Y_0 - I - \frac{w_0}{p_0}N_0 + \frac{1}{1+r}\left(Y_1 + K_1 - \delta K_1 - \frac{w_1}{p_1}N_1\right) = Y_0 - I - \frac{w_0}{p_0}N_0 + \frac{1}{1+r}(K_1 + rK_1) = \dots$$

$$\dots = Y_0 - K_1 + (1-\delta)K_0 - \frac{w_0}{p_0}N_0 + K_1 = K_0 + Y_0 - \delta K_0 - \frac{w_0}{p_0}N_0$$

The capital yield at $t=1$ is rK_1. Because at $t=0$ the capital is assumed to be already remunerated, $Y_0 = \delta K_0 + \frac{w_0}{p_0}N_0$ is fulfilled, and $\frac{V_0}{p_0} = K_0$, that is, the stock market value of the firm coincides with that of the capital stock. Then, the real wealth of households becomes $\mathbf{W} = K_0 + \frac{w_0}{p_0}\mathbf{N_0} + \frac{1}{1+r}\frac{w_1}{p_1}\mathbf{N_1} - G_0 - \frac{G_1}{1+r} = K_0 + \mathbf{H} - \mathbf{G}$, where $\mathbf{H} = \frac{w_0}{p_0}\mathbf{N_0} + \frac{1}{1+r}\frac{w_1}{p_1}\mathbf{N_1}$ is the present value of the human capital of the potentially active population, and $\mathbf{G} = G_0 + \frac{G_1}{1+r}$ is the present value of public spending coinciding with the present value of taxes minus the public debt at the beginning of $t=0$ (see equation (6)).

5. Simultaneous equilibrium of the labour and goods markets

Returning to the aggregate functions found above, from the households, we have that $H_t = H_t\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; \mathbf{N_0}, \mathbf{N_1}, G_0, G_1\right)$ with $H_t = C_0, N_0^s, C_1, N_1^s$, where both the current and future populations and the public spending are exogenous. For firms, we have $N_0^d = N_0^d\left(\frac{w_0}{p_0}\right)$, $I = I\left(\frac{w_1}{p_1}, r\right)$, $Y_0^s = Y_0^s\left(\frac{w_0}{p_0}\right)$, $N_1^d = N_1^d\left(\frac{w_1}{p_1}, r\right)$, and $Y_1^s = Y_1^s\left(\frac{w_1}{p_1}, r\right)$.

The simultaneous equilibrium of the four markets implies satisfying the following system:

$$\begin{cases} N_0^s\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; \mathbf{N_0}, \mathbf{N_1}, G_0, G_1\right) = N_0^d\left(\frac{w_0}{p_0}\right) \\ N_1^s\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; \mathbf{N_0}, \mathbf{N_1}, G_0, G_1\right) = N_1^d\left(\frac{w_1}{p_1}, r\right) \\ Y_0^s\left(\frac{w_0}{p_0}\right) = C_0\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; \mathbf{N_0}, \mathbf{N_1}, G_0, G_1\right) + I\left(\frac{w_1}{p_1}, r\right) + G_0 \end{cases} \quad (8)$$

The fourth market, the goods market at $t=1$, is in equilibrium if all the other markets are as well (Walras' law).

The analysis of the comparative statics can be analysed around the equilibrium of the system (8) by using the linear approximation $Adp_r + Bdx = 0$, with A and B the partial derivative matrices of the system (8)

$$A=\begin{bmatrix} \dfrac{\partial N_0^s}{\partial r} & \dfrac{\partial N_0^s}{\partial\left(\dfrac{w_0}{p_0}\right)}-\dfrac{\partial N_0^d}{\partial\left(\dfrac{w_0}{p_0}\right)} & \dfrac{\partial N_0^s}{\partial\left(\dfrac{w_1}{p_1}\right)} \\ \dfrac{\partial N_1^s}{\partial r}-\dfrac{\partial N_1^d}{\partial r} & \dfrac{\partial N_1^s}{\partial\left(\dfrac{w_0}{p_0}\right)} & \dfrac{\partial N_1^s}{\partial\left(\dfrac{w_1}{p_1}\right)}-\dfrac{\partial N_1^d}{\partial\left(\dfrac{w_1}{p_1}\right)} \\ \dfrac{\partial C_0}{\partial r}+\dfrac{\partial I}{\partial r} & \dfrac{\partial C_0}{\partial\left(\dfrac{w_0}{p_0}\right)}-\dfrac{\partial Y_0^s}{\partial\left(\dfrac{w_0}{p_0}\right)} & \dfrac{\partial C_0}{\partial\left(\dfrac{w_1}{p_1}\right)}+\dfrac{\partial I}{\partial\left(\dfrac{w_1}{p_1}\right)} \end{bmatrix};$$

$$B=\begin{bmatrix} \dfrac{\partial N_0^s}{\partial \mathbf{N_0}} & \dfrac{\partial N_0^s}{\partial \mathbf{N_1}} & \dfrac{\partial N_0^s}{\partial G_0} & \dfrac{\partial N_0^s}{\partial G_1} \\ \dfrac{\partial N_1^s}{\partial \mathbf{N_0}} & \dfrac{\partial N_1^s}{\partial \mathbf{N_1}} & \dfrac{\partial N_1^s}{\partial G_0} & \dfrac{\partial N_1^s}{\partial G_1} \\ \dfrac{\partial C_0}{\partial \mathbf{N_0}} & \dfrac{\partial C_0}{\partial \mathbf{N_1}} & \dfrac{\partial C_0}{\partial G_0}+1 & \dfrac{\partial C_0}{\partial G_1} \end{bmatrix}$$

and $dp_r=\left[dr,\ d\left(\dfrac{w_0}{p_0}\right),\ d\left(\dfrac{w_1}{p_1}\right)\right]'$ the vector of relative price variations; and $dx=\left[d\mathbf{N_0},\ d\mathbf{N_1},\ dG_0,\ dG_1\right]'$ the vector of exogenous variable variations.

Regarding the signs of the A and B matrices, we have that $sign(A)=\begin{bmatrix} + & + & - \\ ? & - & + \\ - & + & ? \end{bmatrix}$ and $sign(B)=\begin{bmatrix} + & - & - & + \\ - & + & + & - \\ + & + & + & - \end{bmatrix}$.

The ambiguity in the signs of the matrix A lies in the impacts of the interest rate on the labour market at $t=1$ and of future real wage on the goods market at $t=0$. In the first case, an increase in the real interest rate would reduce both the supply and demand of labour in the plane $\left(\dfrac{w_1}{p_1}, N_1\right)$, and therefore it would not be possible to know its effects on $\dfrac{w_1}{p_1}$; accordingly, the sign indeterminacy would transfer to the remaining variables that depend on $\dfrac{w_1}{p_1}$. Without ambiguity, the employment level N_1 would decrease. The second indeterminacy concerns the effect of changes in future real wage on the current goods market. For example, an increase of $\dfrac{w_1}{p_1}$ will reduce investment and increase consumption at $t=0$, and therefore, it is not possible to know the net effect on the demand for goods and, consequently, on the remaining variables.

To avoid these two indeterminacies, in the next section we make two simplifying assumptions. The first assumption supposes that the labour supply is exogenous and, therefore, the link between the interest rate and the labour market due to the labour supply side disappears. The second assumption supposes that the production factors are independent, and therefore variations in future real wage do not affect investment. Both assumptions make intertemporal links be absent in the labour market, while the goods market continues to be built based on this link because of the permanent income consumption function and the investment function.

6. Model assuming exogenous labour supplies and independent production factors

Suppose that the full labour potential is supplied independently of what the current real wage and the real interest rates are. In that case, $l_0,\ l_1=0$, and the budget constraint (3) becomes:

$$\frac{b_0+v_0}{p_0}+\frac{w_0}{p_0}-t_0+\frac{1}{1+r}\left(\frac{w_1}{p_1}-t_1\right)\geq c_0+\frac{c_1}{1+r} \qquad (3')$$

where the real wealth -on the left of the equation- is equal to that of equation (4). The difference now is that no wealth is going to be allocated to leisure. In the Appendix A.3, the consumption demand functions are obtained for a specific additive-type utility function similar to that of Section 2 but in which leisure time is absent.

By aggregating for all households the budget constraints (3'), an aggregate real wealth equal to that in equation (5) is obtained. The aggregate consumption functions are $C_0=k_c\mathbf{W}$ and $C_1=k_c\beta(1+r)\mathbf{W}$. Now $k_c=\dfrac{1}{1+\beta}$. The labour supply coincides with the potentially active population of each period, $N_0^s=\sum_i n_{i,0}=\mathbf{N_0}$ and $N_1^s=\sum_i n_{i,1}=\mathbf{N_1}$.

The aggregate wealth is allocated to the consumptions in accordance to the k_c and $k_c\beta$ percentages, and therefore $\mathbf{W}=C_0+\dfrac{1}{1+r}C_1=k_c\mathbf{W}+\dfrac{1}{1+r}k_c\beta(1+r)\mathbf{W}=k_c(1+\beta)\mathbf{W}$.

The derivatives and signs of the aggregate consumption functions are the same as those in section 2. The aggregate saving function is simplified being

$S_0=\left(\dfrac{w_0}{p_0}-t_0\right)\mathbf{N_0}-C_0=\left(\dfrac{w_0}{p_0}-t_0\right)\mathbf{N_0}-k_c\mathbf{W}$, with the derivatives $\dfrac{\partial S_0}{\partial r}=\mathbf{N}_1k_c\dfrac{1}{(1+r)^2}\left(\dfrac{w_1}{p_1}-t_1\right)>0$,

$\dfrac{\partial S_0}{\partial\left(\dfrac{w_0}{p_0}\right)}=\mathbf{N_0}(1-k_c)>0$, $\dfrac{\partial S_0}{\partial\left(\dfrac{w_1}{p_1}\right)}=-\mathbf{N_1}k_c\dfrac{1}{1+r}<0$,

$\frac{\partial S_0}{\partial \mathbf{N_0}} = \left(\frac{w_0}{p_0} - t_0\right)(1-k_c) > 0$, and

$\frac{\partial S_0}{\partial \mathbf{N}_1} = -k_c \frac{1}{1+r}\left(\frac{w_1}{p_1} - t_1\right) < 0$.

As in section 2, it is noted here that a greater future population reduces saving because it increases wealth. The signs found here can be generalized if the functional form of the utility meets the assumptions necessary for the predominance of the sign of the substitution effect on the signs of the income and wealth effects. The consideration of the Ricardian equivalence leaves the wealth equal to that of equation (7). The signs of the consumption and saving functions are the same of those of Section 2. For its part, if the production factors are independent, the cross derivative $F''_{K_1N_1} = 0$, and $\frac{\partial N_1^d}{\partial r} = \frac{\partial I}{\partial\left(\frac{w_1}{p_1}\right)} = 0$.

The simultaneous equilibrium of goods and labour markets considers the following aggregate functions: for households, $C_t = C_t\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; \mathbf{N_0}, \mathbf{N_1}, G_0, G_1\right)$ with $t = 0,\ 1$; and for firms, $N_0^d = N_0^d\left(\frac{w_0}{p_0}\right)$, $I = I(r)$, $Y_0^s = Y_0^s\left(\frac{w_0}{p_0}\right)$, $N_1^d = N_1^d\left(\frac{w_1}{p_1}\right)$ and $Y_1^s = Y_1^s\left(\frac{w_1}{p_1}, r\right)$. Simultaneous equilibrium implies that:

$$\begin{cases} \mathbf{N_0} = N_0^d\left(\frac{w_0}{p_0}\right) \\ \mathbf{N_1} = N_1^d\left(\frac{w_1}{p_1}\right) \\ Y_0^s\left(\frac{w_0}{p_0}\right) = C_0\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; \mathbf{N_0}, \mathbf{N_1}, G_0, G_1\right) + I(r) + G_0 \end{cases} \tag{9}$$

The comparative statics is analysed around an equilibrium through the linear approximation $A_1 dp_r + B_1 dx = 0$, with the partial derivative matrices of the system (9), A_1 and B_1, being

$$A_1 = \begin{bmatrix} 0 & -\frac{\partial N_0^d}{\partial\left(\frac{w_0}{p_0}\right)} & 0 \\ 0 & 0 & -\frac{\partial N_1^d}{\partial\left(\frac{w_1}{p_1}\right)} \\ \frac{\partial C_0}{\partial r} + \frac{\partial I}{\partial r} & \frac{\partial C_0}{\partial\left(\frac{w_0}{p_0}\right)} - \frac{\partial Y_0^s}{\partial\left(\frac{w_0}{p_0}\right)} & \frac{\partial C_0}{\partial\left(\frac{w_1}{p_1}\right)} \end{bmatrix};$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\partial C_0}{\partial \mathbf{N_0}} & \frac{\partial C_0}{\partial \mathbf{N_1}} & \frac{\partial C_0}{\partial G_0} + 1 & \frac{\partial C_0}{\partial G_1} \end{bmatrix}$$

Compared to matrix A, in matrix A_1 the derivatives with unknown sign have been cancelled, resulting now $sign(A_1) = \begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ - & + & + \end{bmatrix}$. In $sign(B_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ + & + & + & - \end{bmatrix}$, it is verified that increases in population are fully transmitted to the labour supplies in the period in which they occur (1s in the first two columns of B_1): wealth effects caused by population changes do not affect leisure time demands. The variations in public spending, which are equivalent to variations in taxation, do not affect labour supplies, which does not seem empirically admissible; neither does the fact that the employment levels vary exclusively because of population variations. Once labour supply has increased after an increase in population, real wage of the corresponding period reduces moving along the labour demand.

The signs on the relative prices are obtained from $dp_r = -A_1^{-1} B_1 dx$, resulting in

$$\begin{bmatrix} dr \\ d\left(\frac{w_0}{p_0}\right) \\ d\left(\frac{w_1}{p_1}\right) \end{bmatrix} = \begin{bmatrix} ? & ? & + & - \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix} \begin{bmatrix} d\mathbf{N_0} \\ d\mathbf{N_1} \\ dG_0 \\ dG_1 \end{bmatrix}.$$

The effects of the variations of the present or future population on the real interest rate are uncertain. For example, if the population increases, then the real wage of the respective period decreases. For the first cause (population increases), consumption increases; for the second cause (real wage decreases), consumption decreases. In the (r, Y) plane, it is not possible to know the net effect on the interest rate given this indeterminacy on the demand for goods. Concerning fiscal expansion, if this is current, then the interest rate increases (crowding-out effect), and it decreases if this occurs in the future. The different sign is because the current fiscal expansion represent a net increase in the demand for goods, given that $0 < \frac{\partial C_0}{\partial G_0} + 1 < 1$, $\frac{\partial C_0}{\partial G_0} = -k_c$, and $0 < k_c < 1$.

Due to the absence of a full parameterization, it is not possible to know the consumption, saving and investment signs resulting from a change in the population sizes because of their uncertain effects, as noted immediately above, on the interest rate. However, it is possible to know that the current production will increase if the current population increases because the real wage would decrease, and that the future population will not affect it. Concerning current fiscal expansion, this reduces consumption, increases savings and reduces investment (crowding-out effect) due to the increase of the interest rate. Future fiscal expansion, by reducing the interest rate, acts in the opposite direction. Summarizing, the absence of intertemporal links in the labour markets, given the simplifying assumptions on exogenous labour supplies and independent factors of production, has been able to determine the signs in the matrix A. However intertemporal links continue to be present through wealth effects in the consumption function, which constitutes a theoretical inconsistency: Why intertemporal links are important in the goods markets but are ignored in the labour market? Of no less importance, indeterminacy of the interest rate variation after changes in population sizes can be hidden by ignoring this component of wealth as usual in short and medium term macroeconomics, but this practice contradicts the mere presence of a labour market in the model.

7. Effects of technological improvements

Suppose that the aggregate production function is affected by neutral technological progress, such that $Y = ZF(K,N)$, and that the variable representative of the "Solow residual", Z, increases. This will affect the labour and investment demands due to the productivity improvement in both factors. In the Appendix A.4, it is proved that both factor demands and product supplies depend positively on technological improvements given the prices of the factors of production. Assuming independent factors of production and exogenous labour supplies, the general equilibrium implies that:

$$\begin{cases} \mathbf{N_0} = N_0^d\left(\frac{w_0}{p_0}; Z_0\right) \\ \mathbf{N_1} = N_1^d\left(\frac{w_1}{p_1}; Z_1\right) \\ Y_0^s\left(\frac{w_0}{p_0}; Z_0\right) = C_0\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; \mathbf{N_0}, \mathbf{N}_1, G_0, G_1\right) + I\left(r; Z_1\right) + G_0 \end{cases}$$

Differentiating, the approximation $A_2 dp_r + B_2 dz = 0$ is obtained, with

$$A_2 = \begin{bmatrix} 0 & \frac{\partial N_0^d}{\partial\left(\frac{w_0}{p_0}\right)} & 0 \\ 0 & 0 & \frac{\partial N_1^d}{\partial\left(\frac{w_1}{p_1}\right)} \\ \frac{\partial C_0}{\partial r} + \frac{\partial I}{\partial r} & \frac{\partial C_0}{\partial\left(\frac{w_0}{p_0}\right)} - \frac{\partial Y_0^s}{\partial\left(\frac{w_0}{p_0}\right)} & \frac{\partial C_0}{\partial\left(\frac{w_1}{p_1}\right)} \end{bmatrix};$$

$$B_2 = \begin{bmatrix} \frac{\partial N_0}{\partial Z_0} & 0 \\ 0 & \frac{\partial N_1}{\partial Z_1} \\ -\frac{\partial Y_0^s}{\partial Z_0} & \frac{\partial K_1}{\partial Z_1} \end{bmatrix}$$

and $dz = \left[dZ_0 \quad dZ_1\right]'$, with $sign(A_2) = \begin{bmatrix} 0 & - & 0 \\ 0 & 0 & - \\ - & + & + \end{bmatrix}$ and $sign(B_2) = \begin{bmatrix} + & 0 \\ 0 & + \\ - & + \end{bmatrix}$. The signs of the relative prices are obtained from $dp_r = -A_2^{-1} B_2 dz$, resulting in the following:

$$\begin{bmatrix} dr \\ d\left(\frac{w_0}{p_0}\right) \\ d\left(\frac{w_1}{p_1}\right) \end{bmatrix} = \begin{bmatrix} ? & + \\ + & 0 \\ 0 & + \end{bmatrix} \begin{bmatrix} dZ_0 \\ dZ_1 \end{bmatrix}$$

Technological improvements increase the labour demand and the real wage of the period in which they occur, but not the employment levels because of the assumption on rigid labour supplies. This seems to contradict empirical experience concerning technological progress: both production and employment increase at least at a macroeconomic level. A current technological improvement has ambiguous effects on the interest rate because i) the increase in the real wage increases consumption, which increases the interest rate in the (r, Y_0) plane; and ii) production increases, which reduces the interest rate. Conversely, future technological improvement increases the interest rate, by increasing the marginal productivity of capital, increases present investment, and consumption because the future real wage increases as well. However, employment, neither present nor future, increases.

8. Conclusions

The model developed here sets the microfoundations of the labour supply and consumption demand decisions of households inside an intertemporal framework of two periods where wealth, real interest rate and real wages bind decisions taken in the four markets considered. The endogeneity of the labour supply decisions make them dependent on wealth, and so on the real interest rate, as well as on the real wages of both periods. For firms, capital is assumed to be given in the short term and, therefore, the investment decision is truly intertemporal. The non-zero cross marginal productivity of factors of production causes the interrelationship between the labour demand at $t=1$ and the investment demand at $t=0$ through the presence of the real wage and the real interest rate in both demands. When considering general equilibrium of the four markets, the presence of the real interest rate in the decision functions of households and firms prevents from obtaining unambiguous conclusions regarding the responses to exogenous variations.
When we assume rigid labour supplies and independent factors of production, the system allows the determinacy of the signs of the matrix of endogenous responses. Nevertheless, the independency of the factors of production seems to contradict empirical experience: investment and employment are positively correlated throughout the business cycle, which is coherent with the complementarity assumption of factors. Some, but not all, macroeconomic textbooks establishes an investment function depending exclusively on the interest rate, and not on the real wage, thus implicitly accepting the independency assumption. Independency of factors and rigid labour supplies make intertemporal links be absent in the labour market, while the goods market continues to be built based on this link because of the permanent income consumption function. Once again, empirical and theoretical coherence seem to require the dependency of the labour supplies both on the real wage, present and future, and on the real interest rate: when the future has been introduced into a decision problem, interest rate is indispensable. This cannot be considered in one market and not in the other when the same agent is taking simultaneous decisions in all the markets.

A second relevant aspect has been introduced in this work, that is, the relation between macromagnitudes and population levels. We have obtained the aggregate magnitudes from homogeneous agents in a very simple way. This has made it possible to consider the population, current and future, as a scale variable of the macroeconomic aggregates and as a component of the domestic wealth. The latter is usually absent from the models being used. An increase in population increases every aggregate, as expected, except concerning an increase in the future population, which reduces saving, what is consistent with its effects on current consumption, because it implies an increase of wealth. When labour decisions are wealth-based, not every increase of population implies an increase in labour supply: some portion of the new wealth serves to demand more leisure time. This behaviour is corroborated with the increase in leisure time (working hours, retirement age) observed in wealthier economies.

Finally, an analysis on the effects of technological improvements is conducted in line with the models of the Real Business Cycle theory that attempts to attribute the cyclical dynamic to technological shocks in economies without money, money thereby becoming irrelevant for explaining the cycle. Future technological shocks increase the real interest rate; however, current technological shocks have an ambiguous effect. The positive correlation between employment, investment and output points again to the need for assuming complementarity of production factors also in presence of technology shocks. We have not considered in this work the essential question on the monetary origin of the business cycle. We have confined our analysis to a real economy in which the interest rate, as an intertemporal link, introduces interesting indeterminacies. Money is "the other" intertemporal link closely related to the real interest rate by means of expected inflation and the yield of public debt and real assets, which must be present in macroeconomic modelling (Cendejas *et al.*, 2014).

This study shows the difficulty of using general interdependence models to obtain unambiguous conclusions. In our case, the disappearance of sign indeterminacies is achieved by imposing very strict assumptions that contradict empirical evidence. Another possibility, not addressed here, is the use of alternative parameterizations that would resolve the indeterminacies of signs in one sense or another. This is the strategy typically followed in the calibration of models. In that case, to limit the large set of possible parametric combinations, certain values obtained from a previous econometric estimation are imposed. Despite this currently widespread procedure, the various parameterizing possibilities must be borne in mind and need to be reasonably explored to verify how robust the signs obtained are. In summary, the attempt to build macroeconomics on a more rigorous microeconomic basis to avoid the usual dichotomies (short term vs. long term, real vs. monetary) does not yet seem to have been settled by a fully coherent proposal.

Appendixes

A.1. Maximization problem of Section 2

We assume the following additive-type utility function:

$$U = U\left(c_0, l_0, c_1, l_1\right) = u(c_o) + \psi(l_0) + \beta\left(u(c_1) + \psi(l_1)\right)$$

where $0 < \beta < 1$ is the discount factor. The $u(.)$ and $\psi(.)$ functions are such that $u' > 0$ and $\psi' > 0$ (no saturation) and $u'' < 0$ and $\psi'' < 0$ (strict concavity). The budget constraint is that of equation (3)

$$W\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; t_0, t_1\right) \geq \frac{w_0}{p_0} l_0 + \frac{1}{1+r}\frac{w_1}{p_1} l_1 + c_0 + \frac{c_1}{1+r} \tag{3}$$

The Lagrangian of the optimization problem for the representative household becomes the following:

$$L = u(c_0) + \psi(l_0) + \beta\left(u(c_1) + \psi(l_1)\right) + \lambda\left(W(.) - \frac{w_0}{p_0} l_0 - \frac{1}{1+r}\frac{w_1}{p_1} l_1 - c_0 - \frac{1}{1+r} c_1\right)$$

whose first-order conditions (FOCs) are as follows:

$$\begin{cases} \dfrac{\partial L}{\partial c_0} = u'(c_0) - \lambda = 0 \\ \dfrac{\partial L}{\partial c_1} = \beta u'(c_1) - \lambda \dfrac{1}{1+r} = 0 \\ \dfrac{\partial L}{\partial l_0} = \psi'(l_0) - \lambda \dfrac{w_0}{p_0} = 0 \\ \dfrac{\partial L}{\partial l_1} = \beta\psi'(l_1) - \lambda \dfrac{1}{1+r}\dfrac{w_1}{p_1} = 0 \\ \dfrac{\partial L}{\partial \lambda_0} = W(.) - \dfrac{w_0}{p_0} l_0 - \dfrac{1}{1+r}\dfrac{w_1}{p_1} l_1 - c_0 - \dfrac{1}{1+r} c_1 = 0 \end{cases} \Rightarrow \begin{cases} \beta u'(c_1) = u'(c_0)\dfrac{1}{1+r} \\ \psi'(l_0) = u'(c_0)\dfrac{w_0}{p_0} \\ \psi'(l_1) = u'(c_1)\dfrac{w_1}{p_1} \\ \beta\psi'(l_1) = u'(c_0)\dfrac{1}{1+r}\dfrac{w_1}{p_1} \\ W(.) = \dfrac{w_0}{p_0} l_0 + \dfrac{1}{1+r}\dfrac{w_1}{p_1} l_1 + c_0 + \dfrac{1}{1+r} c_1 \end{cases}$$

which ensures that the following equalities between marginal rates of substitution (MRS) and relative prices are satisfied:

$MRS_{c_o,c_1} = \dfrac{u'(c_o)}{\beta u'(c_1)} = 1+r;\ MRS_{l_t,c_t} = \dfrac{\psi'(l_t)}{u'(c_t)} = \dfrac{w_t}{p_t}$ with $t = 0,1$;

$MRS_{c_0,l_1} = \dfrac{u'(c_0)}{\beta\psi'(l_1)} = (1+r)\dfrac{p_1}{w_1}$; and $MRS_{l_0,c_1} = \dfrac{\psi'(l_0)}{\beta u'(c_1)} = (1+r)\dfrac{w_0}{p_0}$.

The hypothesis of the implicit functions theorem (Barbolla and Sanz, 1995), here fulfilled, ensure that the demand functions, $h_t = h_t\left(r, \dfrac{w_0}{p_0}, \dfrac{w_1}{p_1}; t_0, t_1\right)$ with $h_t = c_o, c_1, l_0, l_1$, exist and are differentiable.

To obtain explicit expressions of the resulting demand functions, we assume the following utility function:

$U = U\left(c_0, l_0, c_1, l_1\right) = \ln c_0 + \psi \ln l_0 + \beta\left(\ln c_1 + \psi \ln l_1\right)$ with $0 < \beta < 1$ and $\psi > 0$.

The first three FOCs are the following:

$$\begin{cases} \dfrac{1}{c_0} = \beta(1+r)\dfrac{1}{c_1} \\ \psi\dfrac{c_0}{l_0} = \dfrac{w_0}{p_0} \\ \psi\dfrac{c_1}{l_1} = \dfrac{w_1}{p_1} \end{cases} \Rightarrow \begin{cases} c_1 = \beta(1+r)c_0 \\ l_0 = \psi\dfrac{p_0}{w_0}c_0 \\ l_1 = \psi\dfrac{p_1}{w_1}c_1 = \psi\dfrac{p_1}{w_1}\beta(1+r)c_0 \end{cases}$$

By replacing the decision variables, written as a function of c_0, in the budget constraint

$$W(.) = \frac{w_0}{p_0}\psi\frac{p_0}{w_0}c_0 + \frac{1}{1+r}\frac{w_1}{p_1}\psi\frac{p_1}{w_1}\beta(1+r)c_0 + c_0 + \frac{1}{1+r}\beta(1+r)c_0 = \left(1+\psi+\beta(1+\psi)\right)c_0$$

we obtain the wealth-dependent consumption function (consumption as permanent income, Friedman, 1957), $c_0 = k_c W$, with $k_c = \dfrac{1}{1+\psi+\beta(1+\psi)}$ the marginal propensity to consume wealth. Accordingly, the remaining demand functions are obtained: $c_1 = k_c\beta(1+r)W$, $l_0 = k_c\psi\dfrac{p_0}{w_0}W$, and $l_1 = k_c\beta\psi\dfrac{p_1}{w_1}(1+r)W$. It is verified that wealth is allocated between the four goods (l_0, l_1, c_0, c_1) in accordance with the percentages $k_c\psi$, $k_c\beta\psi$, k_c and $k_c\beta$, respectively. To see this, it is sufficient to substitute in the respective FOC:

$$W = \frac{w_0}{p_0}k_c\psi\frac{p_0}{w_0}W + \frac{1}{1+r}\frac{w_1}{p_1}k_c\beta\psi\frac{p_1}{w_1}(1+r)W + k_cW + \frac{1}{1+r}k_c\beta(1+r)W = k_c(1+\psi+\beta(1+\psi))W$$

where $k_c + k_c\beta + k_c\psi + k_c\beta\psi = 1$.

The signs of the consumption demand functions, current and future, are as follows (assuming $\dfrac{w_0}{p_0} > t_0$, $\dfrac{w_1}{p_1} > t_1$, and $\dfrac{b_0+v_0}{p_0} > 0$):

$$\frac{\partial c_0}{\partial r} = -k_c\frac{1}{(1+r)^2}\left(\frac{w_1}{p_1} - t_1\right) < 0; \qquad \frac{\partial c_0}{\partial\left(\dfrac{w_0}{p_0}\right)} = k_c > 0; \qquad \frac{\partial c_0}{\partial\left(\dfrac{w_1}{p_1}\right)} = \frac{k_c}{1+r} > 0;$$

$$\frac{\partial c_1}{\partial r} = k_c\beta\left(\frac{b_0+v_0}{p_0} + \frac{w_0}{p_0} - t_0\right) > 0; \quad \frac{\partial c_1}{\partial\left(\dfrac{w_0}{p_0}\right)} = k_c\beta(1+r) > 0; \quad \frac{\partial c_1}{\partial\left(\dfrac{w_1}{p_1}\right)} = k_c\beta > 0.$$

Additionally, the signs of the leisure demands are:

$$\frac{\partial l_0}{\partial r} = -k_c\psi\frac{1}{(1+r)^2}\frac{p_0}{w_0}\left(\frac{w_1}{p_1} - t_1\right) < 0; \quad \frac{\partial l_0}{\partial\left(\dfrac{w_0}{p_0}\right)} = k_c\psi\left(\frac{p_0}{w_0}\right)^2\left(\frac{w_0}{p_0} - W\right) < 0; \quad \frac{\partial l_0}{\partial\left(\dfrac{w_1}{p_1}\right)} = k_c\psi\frac{p_0}{w_0}\frac{1}{1+r} > 0;$$

$$\frac{\partial l_1}{\partial r} = k_c\beta\psi\frac{p_1}{w_1}\left(W - \frac{1}{1+r}\left(\frac{w_1}{p_1} - t_1\right)\right) > 0; \quad \frac{\partial l_1}{\partial\left(\dfrac{w_0}{p_0}\right)} = k_c\beta\psi\frac{p_1}{w_1}(1+r) > 0; \quad \frac{\partial l_1}{\partial\left(\dfrac{w_1}{p_1}\right)} = k_c\beta\psi(1+r)\left(\frac{p_1}{w_1}\right)^2\left(\frac{1}{1+r}\frac{w_1}{p_1} - W\right) < 0.$$

The real saving of a representative household is the unconsumed disposible income.

$$s_0 = \frac{w_0}{p_0} n_0 - t_0 - c_0 = \frac{w_0}{p_0} - t_0 - \frac{w_0}{p_0} k_c \psi \frac{p_0}{w_0} W - k_c W = \frac{w_0}{p_0} - t_0 - k_c (1+\psi) W$$

whose signs are the following:

$$\frac{\partial s_0}{\partial r} = k_c (1+\psi) \frac{1}{(1+r)^2} \left(\frac{w_1}{p_1} - t_1 \right) > 0; \quad \frac{\partial s_0}{\partial \left(\frac{w_0}{p_0} \right)} = 1 - k_c (1+\psi) > 0; \quad \frac{\partial s_0}{\partial \left(\frac{w_1}{p_1} \right)} = -k_c (1+\psi) \frac{1}{1+r} < 0.$$

A.2. Maximization problem of Section 4

A representative firm maximizes its present value

$$VA(y_0, y_1, k_1, n_0, n_1) = p_0 y_0 - p_0 I_0 - w_0 n_0 + \frac{1}{1+i} \left(p_1 y_1 + p_1 k_1 - \delta p_1 k_1 - w_1 n_1 \right)$$

subject to $f(\bar{k}_0, n_0) \geq y_0$, $f(k_1, n_1) \geq y_1$ and $I_0 = k_1 - (1-\delta)\bar{k}_0$, where the notation and the assumptions of the production function are detailed in Section 4.

From the Lagrangian

$$L = p_0 y_0 - p_0 k_1 + (1-\delta) p_0 \bar{k}_0 - w_0 n_0 + \frac{1}{1+i} \left(p_1 y_1 + p_1 k_1 - \delta p_1 k_1 - w_1 n_1 \right) + \lambda_0 \left(f(\bar{k}_0, n_0) - y_0 \right) + \lambda_1 \left(f(k_1, n_1) - y_1 \right)$$

FOCs can be obtained leading to product supply and factor demand functions that are continuous and differentiable:

$$\left\{ \begin{array}{c} \dfrac{\partial L}{\partial y_0} = p_0 - \lambda_0 = 0 \\ \dfrac{\partial L}{\partial y_1} = p_1 - \lambda_1 = 0 \\ \dfrac{\partial L}{\partial n_0} = -w_0 + \lambda_0 f'_{n_0}(\bar{k}_0, n_0) = 0 \\ \dfrac{\partial L}{\partial n_1} = \dfrac{1}{1+i} \left(-w_1 + f'_{n_1}(k_1, n_1) \right) = 0 \\ \dfrac{\partial L}{\partial k_1} = -p_0 + \dfrac{1}{1+i} \left(p_1 - \delta p_1 + p_1 f'_{k_1}(k_1, n_1) \right) = 0 \\ \dfrac{\partial L}{\partial \lambda_0} = f(\bar{k}_0, n_0) - y_0 = 0 \\ \dfrac{\partial L}{\partial \lambda_1} = f(k_1, n_1) - y_1 = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{c} p_0 f'_{n_0}(\bar{k}_0, n_0) = w_0 \\ p_1 f'_{n_1}(k_1, n_1) = w_1 \\ p_1 \left(1 - \delta + f'_{k_1}(k_1, n_1) \right) = p_0 (1+i) \\ f(\bar{k}_0, n_0) = y_0 \\ f(k_1, n_1) = y_1 \end{array} \right.$$

The zero-degree homogeneity in prices of the product supply and factors' demand functions makes it possible to express these, equivalently, as a function of the monetary prices as $h_t = h_t(p_0, p_1, w_0, w_1, i)$ or also as a function of the relative prices as $h_t = h_t\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1} \right)$ with $h_t = n_0, n_1, k_1, y_0, y_1$. Thus, for the labour demand, $f'_n(k_t, n_t) = \frac{w_t}{p_t}$ with $t = 0,\ 1$. For the capital demand at $t = 1$, $1 - \delta + f'_{k_1}(k_1, n_1) = \frac{p_0}{p_1}(1+i) = \frac{1+i}{1+\pi} = 1 + r$, that leads to $f'_{k_1}(k_1, n_1) = r + \delta$.

The signs of the factor demand functions are obtained by differentiating the FOCs as follows:

$$\begin{cases} f'_{n_0} = \dfrac{w_0}{p_0} \\ f'_{n_1} = \dfrac{w_1}{p_1} \\ f'_{k_1} = r+\delta \end{cases} \Rightarrow \begin{cases} f''_{n_0n_0}dn_0 = d\left(\dfrac{w_0}{p_0}\right) \\ f''_{k_1n_1}dk_1 + f''_{n_1n_1}dn_1 = d\left(\dfrac{w_1}{p_1}\right) \\ f''_{k_1k_1}dk_1 + f''_{k_1n_1}dn_1 = dr \end{cases} \Rightarrow \begin{bmatrix} f''_{n_0n_0} & 0 & 0 \\ 0 & f''_{n_1n_1} & f''_{k_1n_1} \\ 0 & f''_{k_1n_1} & f''_{k_1k_1} \end{bmatrix} \begin{bmatrix} dn_0 \\ dn_1 \\ dk_1 \end{bmatrix} = \begin{bmatrix} d\left(\dfrac{w_0}{p_0}\right) \\ d\left(\dfrac{w_1}{p_1}\right) \\ dr \end{bmatrix}$$

Accordingly,

$$\begin{bmatrix} dn_0 \\ dn_1 \\ dk_1 \end{bmatrix} = \frac{1}{f''_{n_0n_0}f''_{k_1k_1}f''_{n_1n_1} - f''_{n_0n_0}f''^2_{k_1n_1}} \begin{bmatrix} f''_{k_1k_1}f''_{n_1n_1} - f''^2_{k_1n_1} & 0 & 0 \\ 0 & f''_{n_0n_0}f''_{k_1k_1} & -f''_{n_0n_0}f''_{k_1n_1} \\ 0 & -f''_{n_0n_0}f''_{k_1n_1} & f''_{n_0n_0}f''_{n_1n_1} \end{bmatrix} \begin{bmatrix} d\left(\dfrac{w_0}{p_0}\right) \\ d\left(\dfrac{w_1}{p_1}\right) \\ dr \end{bmatrix}$$

with $f''_{k_1k_1}f''_{n_1n_1}f''_{n_0n_0} - f''_{n_0n_0}f''^2_{k_1n_1} = f''_{n_0n_0}\left(f''_{k_1k_1}f''_{n_1n_1} - f''^2_{k_1n_1}\right) = f''_{n_0n_0}\left|Hf(k_1,n_1)\right| < 0$ due to the hypothesis of strict concavity. Simplifying $dn_0 = \dfrac{1}{f''_{n_0n_0}}d\left(\dfrac{w_0}{p_0}\right)$, $dn_1 = \dfrac{1}{\left|Hf(k_1,n_1)\right|}\left(-f''_{k_1n_1}dr + f''_{k_1k_1}d\left(\dfrac{w_1}{p_1}\right)\right)$ and $dk_1 = \dfrac{1}{\left|Hf(k_1,n_1)\right|}\left(f''_{n_1n_1}dr - f''_{k_1n_1}d\left(\dfrac{w_1}{p_1}\right)\right)$.

The signs are as follows. For the labour demand for $t=0$, $\dfrac{\partial n_0}{\partial\left(\dfrac{w_0}{p_0}\right)} = \dfrac{1}{f''_{n_0n_0}} < 0$. For the factor demand for $t=1$, $\dfrac{\partial k_1}{\partial r} = \dfrac{f''_{n_1n_1}}{\left|Hf(k_1,n_1)\right|} < 0$ and $\dfrac{\partial n_1}{\partial\left(\dfrac{w_1}{p_1}\right)} = \dfrac{f''_{k_1k_1}}{\left|Hf(k_1,n_1)\right|} < 0$. The cross-effect sign depends on the sign of $f''_{k_1n_1}$. If the factors are complementary, $f''_{k_1n_1} > 0$ and $\dfrac{\partial k_1}{\partial\left(\dfrac{w_1}{p_1}\right)} = \dfrac{\partial n_1}{\partial r} = \dfrac{-f''_{k_1n_1}}{\left|Hf(k_1,n_1)\right|} < 0$.

The product supply function for $t=0$ is obtained from $dy_0 = f'_{n_0}dn_0$, and therefore, $\dfrac{\partial y_0}{\partial\left(\dfrac{w_0}{p_0}\right)} = \dfrac{f'_{n_0}}{f''_{n_0n_0}} < 0$, $\dfrac{\partial y_0}{\partial p_0} = \dfrac{-w_0}{{p_0}^2}\dfrac{f'_{n_0}}{f''_{n_0n_0}} > 0$ and $\dfrac{\partial y_0}{\partial w_0} = \dfrac{1}{p_0}\dfrac{f'_{n_0}}{f''_{n_0n_0}} < 0$. For the production of $t=1$, accepting the complementary factors assumption, $dy_1 = f'_{k_1}dk_1 + f'_{n_1}dn_1$ leads to $\dfrac{\partial y_1}{\partial r} = \dfrac{f'_{k_1}f''_{n_1n_1} - f'_{n_1}f''_{k_1n_1}}{\left|Hf(k_1,n_1)\right|} < 0$ and $\dfrac{\partial y_1}{\partial\left(\dfrac{w_1}{p_1}\right)} = \dfrac{f'_{n_1}f''_{k_1k_1} - f'_{k_1}f''_{k_1n_1}}{\left|Hf(k_1,n_1)\right|} < 0$.

If the factors are independent $\frac{\partial k_1}{\partial\left(\frac{w_1}{p_1}\right)} = \frac{\partial n_1}{\partial r} = 0$, then the system of demand functions would be simplified, leaving an exclusive dependency on the price of the own factor $k_1 = k_1(r);\ n_1 = n_1\left(\frac{w_1}{p_1}\right)$; specifically, $\frac{\partial k_1}{\partial r} = \frac{1}{f''_{k_1k_1}} < 0$ and $\frac{\partial n_1}{\partial\left(\frac{w_1}{p_1}\right)} = \frac{1}{f''_{n_1n_1}} < 0$. In the supply functions, the signs do not change, leaving $\frac{\partial y_1}{\partial r} = \frac{f'_{k_1}}{f''_{k_1k_1}} < 0$ and $\frac{\partial y_1}{\partial\left(\frac{w_1}{p_1}\right)} = \frac{f'_{n_1}}{f''_{n_1n_1}} < 0$.

A.3. Maximization problem of Section 6

When no decisions on labour supply are made, the additive-type utility function becomes:

$$U = U(c_0, c_1) = u(c_o) + \beta u(c_1)$$

with $0 < \beta < 1$ being the discount factor. It is verified that $u' > 0$ and $u'' < 0$. Suppose that the full labour potential is supplied independently of which the current real wage and the real interest rates are. In that case, $l_0,\ l_1 = 0$, and the budget constraint (3) becomes:

$$W\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; t_0, t_1\right) \geq c_0 + \frac{c_1}{1+r} \qquad (3')$$

where $W(.)$ is identical to that of equation (4). The difference now is that no quantity of this wealth is going to be allocated to leisure.

The Lagrangian of the problem becomes the following:

$$L = u(c_0) + \beta u(c_1) + \lambda\left(W(.) - c_0 - \frac{1}{1+r}c_1\right)$$

whose FOCs are as follows:

$$\begin{cases} \frac{\partial L}{\partial c_0} = u'(c_0) - \lambda = 0 \\ \frac{\partial L}{\partial c_1} = \beta u'(c_1) - \lambda\frac{1}{1+r} = 0 \\ \frac{\partial L}{\partial \lambda_0} = W(.) - c_0 - \frac{1}{1+r}c_1 = 0 \end{cases} \Rightarrow \begin{cases} \beta u'(c_1) = u'(c_0)\frac{1}{1+r} \\ W(.) = c_0 + \frac{1}{1+r}c_1 \end{cases} \Rightarrow \begin{cases} MRS_{c_o, c_1} = \frac{u'(c_o)}{\beta u'(c_1)} = 1+r \\ W(.) = c_0 + \frac{1}{1+r}c_1 \end{cases}$$

which makes it possible to obtain the system of consumption demand functions $c_t = c_t\left(r, \frac{w_0}{p_0}, \frac{w_1}{p_1}; t_0, t_1\right)$ with $t = 0,1$.

To obtain explicit expressions, we assume the following utility function

$$U = U(c_0, c_1) = \ln c_0 + \beta \ln c_1$$

whose first two FOCs are used to obtain the equality $\frac{1}{c_0} = \beta(1+r)\frac{1}{c_1}$, from which $c_1 = \beta(1+r)c_0$.

By replacing the decision variables in the budget constraint as a function of c_0

$W(.) = c_0 + \frac{1}{1+r}\beta(1+r)c_0 = c_0 + \beta c_0 = (1+\beta)c_0$, the current consumption function $c_0 = \frac{1}{1+\beta}W = k_c W$ is obtained, with $k_c = \frac{1}{1+\beta}$ being the consumed wealth percentage. This function helps to obtain the function for future consumption as $c_1 = k_c\beta(1+r)W$. It is verified that the wealth is distributed in c_0 and c_1 in the k_c and $k_c\beta$ percentages that add up to the unit.

The derivatives and signs of the functions of the current and future consumption demand match those analysed in the maximization problem of Appendix A.1. The saving function is modified as the allocation of wealth to the demand of leisure is not necessary: leisure is not demanded at all. We now have the following:

$s_0 = \frac{w_0}{p_0} - t_0 - c_0 = \frac{w_0}{p_0} - t_0 - k_c W$ with the derivatives $\frac{\partial s_0}{\partial r} = k_c \frac{1}{(1+r)^2}\left(\frac{w_1}{p_1} - t_1\right) > 0$,

$\frac{\partial s_0}{\partial\left(\frac{w_0}{p_0}\right)} = 1 - k_c > 0$ and $\frac{\partial s_0}{\partial\left(\frac{w_1}{p_1}\right)} = -k_c \frac{1}{1+r} < 0$.

A.4. Effects of productivity improvements

By differentiating the FOCs of the aggregate production function $Y = ZF(K,N)$ with respect to Z and the quantities of factors:

$$\begin{cases} Z_0 F'_{N_0} = \frac{w_0}{p_0} \\ Z_1 F'_{N_1} = \frac{w_1}{p_1} \\ Z_1 F'_{K_1} = r + \delta \end{cases} \Rightarrow \begin{cases} Z_0 F''_{N_0N_0} dN_0 + F'_{N_0} dZ_0 = 0 \\ Z_1 F''_{K_1N_1} dK_1 + Z_1 F''_{N_1N_1} dN_1 + F'_{N_1} dZ_1 = 0 \Rightarrow \ldots \\ Z_1 F''_{K_1K_1} dK_1 + Z_1 F''_{K_1N_1} dN_1 + F'_{K_1} dZ_1 = 0 \end{cases}$$

$$\ldots \Rightarrow \begin{bmatrix} Z_0 F''_{N_0N_0} & 0 & 0 \\ 0 & Z_1 F''_{N_1N_1} & Z_1 F''_{K_1N_1} \\ 0 & Z_1 F''_{K_1N_1} & Z_1 F''_{K_1K_1} \end{bmatrix} \begin{bmatrix} dN_0 \\ dN_1 \\ dK_1 \end{bmatrix} = \begin{bmatrix} -F'_{N_0} dZ_0 \\ -F'_{N_1} dZ_1 \\ -F'_{K_1} dZ_1 \end{bmatrix}$$

and inverting

$$\begin{bmatrix} dN_0 \\ dN_1 \\ dK_1 \end{bmatrix} = \frac{1}{Z_0 Z_1^2 F''_{N_0N_0} \left|HF(K_1,N_1)\right|} \begin{bmatrix} Z_1^2(F''_{K_1K_1}F''_{N_1N_1} - F''^2_{K_1N_1}) & 0 & 0 \\ 0 & Z_0 Z_1 F''_{N_0N_0} F''_{K_1K_1} & -Z_0 Z_1 F''_{N_0N_0} F''_{K_1N_1} \\ 0 & -Z_0 Z_1 F''_{N_0N_0} F''_{K_1N_1} & Z_0 Z_1 F''_{N_0N_0} F''_{N_1N_1} \end{bmatrix} \begin{bmatrix} -F'_{N_0} dZ_0 \\ -F'_{N_1} dZ_1 \\ -F'_{K_1} dZ_1 \end{bmatrix}$$

When applying the assumption of independent production factors, $F''_{K_1N_1} = 0$, and the derivatives and signs of the factor demand curves are $\frac{\partial N_0}{\partial Z_0} = \frac{-F'_{N_0}}{Z_0 F''_{N_0N_0}} > 0$, $\frac{\partial N_1}{\partial Z_1} = \frac{-F'_{N_1}}{Z_1 F''_{N_1N_1}} > 0$ and $\frac{\partial K_1}{\partial Z_1} = \frac{-F'_{K_1}}{Z_1 F''_{K_1K_1}} > 0$. So, the demanded quantities of factors increase with technological improvements.

From $Y = ZF(K,N)$, we have that $dY = FdZ + ZdF = \frac{Y}{Z}dZ + Z\left(F_K^{'}dK + F_N^{'}dN\right)$, which allow the obtaining of changes in the supplies of goods: $dY_0 = \frac{Y_0}{Z_0}dZ_0 + Z_0 F_{N_0}^{'} dN_0$ and $dY_1 = \frac{Y_1}{Z_1}dZ_1 + Z_1\left(F_{K_1}^{'}dK_1 + F_{N_1}^{'}dN_1\right)$. Substituting dN_0, dK_1 and dN_1 in them, it is proved that, given the prices of the factors, the effect of technological improvements on product levels is positive: $\frac{\partial Y_0}{\partial Z_0} = \frac{Y_0}{Z_0} - \frac{F_{N_0}^{'2}}{F_{N_0 N_0}^{''}} > 0$ and $\frac{\partial Y_1}{\partial Z_1} = \frac{Y_1}{Z_1} - \frac{F_{K_1}^{'2}}{F_{K_1 K_1}^{''}} - \frac{F_{N_1}^{'2}}{F_{N_1 N_1}^{''}} > 0$.

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